NAME: SOL'NS

Test Prep 2

There is a quick problem on this page. If you finish this page, try the extra problems on the back (same scoring as before: this page is worth all 10 points, but you can get some points back by getting problems correct on the back). You have 10 minutes!

Consider $\frac{dy}{dx} = -\frac{ye^x}{e^x+1}$ with $y(0) = 4$.

1. This is a rare problem where all our methods will work (it's separable, it's linear, and it's exact). Show the first step of each of these methods. That is, rewrite the problem in the correct form for each method. In other words, give me the 6 indicated functions:

$$\frac{1}{y}\,dy = -\frac{e^x}{e^x+1}\,dx$$

in the form $f(y)dy = h(x)dx$: $f(y) = \underline{\frac{1}{y}}$, $h(x) = \underline{-\frac{e^x}{e^x+1}}$

$$\frac{dy}{dx} + \frac{e^x}{e^x+1}y = 0$$

in the form $\frac{dy}{dx} + p(x)y = g(x)$: $p(x) = \underline{\frac{e^x}{e^x+1}}$, $g(x) = \underline{0}$

$$(e^x+1)\frac{dy}{dx} = -ye^x \Rightarrow ye^x + (e^x+1)\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = e^x \quad \frac{\partial N}{\partial x} = e^x$$

SAME!

in the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$: $M(x,y) = \underline{ye^x}$, $N(x,y) = \underline{e^x+1}$

2. Give the **explicit** solution to the differential equation (using any method you like). Remember to use the initial condition to find all constants of integration.

SEPARATING

$$\int \frac{1}{y}dy = \int -\frac{e^x}{e^x+1}dx \qquad u = e^x+1,\ du = e^x dx$$

$$\ln|y| = \int -\frac{1}{u}du \qquad \text{NOTE } u>0$$

$$\ln|y| = -\ln(u) + C \qquad -\ln(u) = \ln\left(\frac{1}{u}\right)$$

$$y = \pm e^{-\ln(u)} \cdot e^C \qquad D = \pm e^C$$

$$y = D\frac{1}{u} = \frac{D}{e^x+1}$$

INTEGRATING FACTOR

$$\int p(x)dx = \int \frac{e^x}{e^x+1}dx \qquad u = e^x+1,\ du = e^x dx$$

$$= \int \frac{1}{u}du$$

$$= \ln(u) + C$$

$$= \ln(e^x+1) + C \qquad C=0$$

$$\mu(x) = e^{\ln(e^x+1)} = e^x+1$$

$$\mu(x)\left[\frac{dy}{dx} + \frac{e^x}{e^x+1}y = 0\right]$$

$$\Rightarrow \frac{d}{dx}\left[(e^x+1)y\right] = 0$$

$$(e^x+1)y = C$$

$$y = \frac{C}{e^x+1}$$

EXACT METHOD

$$\int M(x,y)dx = \int ye^x dx = ye^x + C_1(y)$$

$$\int N(x,y)dy = \int e^x+1\,dy = ye^x + y + C_2(x)$$

$$\Psi(x,y) = ye^x + y$$

$$ye^x + y = C$$

$$\Downarrow$$

$$y(e^x+1) = C$$

$$y = \frac{C}{e^x+1}$$

INITIAL CONDITION $y(0) = 4 \Rightarrow 4 = \frac{C}{e^0+1} \Rightarrow C = 8$

$$\boxed{y(x) = \frac{8}{e^x+1}}$$

Extra Problems

1. Find and classify all equilibrium solutions to $\frac{dy}{dt} = (10-y)^2(y^2-4) = f(y)$

$$= (10-y)^2(y+2)(y-2) \stackrel{?}{=} 0$$

$$\boxed{y(t) = -2,\ y(t) = 2,\ y(t) = 10}$$

$10 < y \Rightarrow f(y)$ POS $+$

10 SEMISTABLE

$2 < y < 10 \Rightarrow f(y)$ POS $+$

2 UNSTABLE

$-2 < y < 2 \Rightarrow f(y)$ NEG $-$

-2 STABLE

$y < -2 \Rightarrow f(y)$ POSITIVE $+$

$y(t) = 10$	SEMISTABLE
$y(t) = 2$	UNSTABLE
$y(t) = -2$	STABLE

2. Consider $\frac{dy}{dt} = (y-2)^{1/5}$ with $y(0) = 2$.
The solution is NOT guaranteed to be unique (you should know why).
Give two different solutions to this differential equation.

BECAUSE $\frac{\partial f}{\partial y} = \frac{1}{5(y-2)^{4/5}}$ IS DISCONTINUOUS AT $y = 2$. AND THE INITIAL CONDITION IS AT $y(0) = 2$.

ONE SOL'N = $\boxed{y(t) = 2}$ ⇐ Equilibrium Sol'n

ANOTHER: $\int \frac{1}{(y-2)^{1/5}}\, dy = \int dt$

$$\frac{5}{4}(y-2)^{4/5} = t + c$$

$$y - 2 = \pm\left(\frac{4}{5}t + \frac{4}{5}c\right)^{5/4}$$

$$y = 2 \pm \left(\frac{4}{5}t + D\right)^{5/4}$$

$$y(0) = 2 \Rightarrow D = 0$$

$$\boxed{y(t) = 2 + \left(\frac{4}{5}t\right)^{5/4}}$$

$$\boxed{y(t) = 2 - \left(\frac{4}{5}t\right)^{5/4}}$$

3 SOL'NS ARE GIVEN HERE

3. Consider $\frac{dy}{dt} = (y-2)^2$ with $y(0) = 2$.
The solution IS guaranteed to be unique (you should also know why).
Give the unique solution.

NO DISCONTINUITY!

SO THE EQUILIBRIUM SOL'N IS THE ONLY SOL'N.

$\boxed{y(t) = 2}$ DONE! NO MORE WORK NEEDED!

ASIDE: IF YOU TRY TO SOLVE YOU GET

$$\int \frac{1}{(y-2)^2}\, dy = \int dt$$

$$-\frac{1}{y-2} = t + c$$

$$\Rightarrow y - 2 = \frac{-1}{t+c}$$

$$y = 2 - \frac{1}{t+c}$$

AND SOL'NS ARE STILL UNIQUE!

IF YOU TRY $y(0) = 2$ THEN YOU GET $2 = 2 - \frac{1}{0+c}$

$$\Rightarrow 0 = -\frac{1}{c}$$

which is not possible

THUS, this does not contain the equilibrium sol'n